On the Kadomtsev–Petviashvili hierarchy, $\hat{W}_\infty$ algebra, and conformal SL(2,*R*)/U(1) model. II. The quantum case

Feng Yu and Yong-Shi Wu
Department of Physics, University of Utah, Salt Lake City, Utah 84112

(Received 7 July 1993; accepted for publication 19 August 1993)

This article is devoted to constructing a quantum version of the famous Kadomtsev–Petviashvili (KP) hierarchy by deforming its second Hamiltonian structure, namely, the nonlinear $\hat{W}_\infty$ algebra. This is achieved by quantizing the conformal noncompact $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ coset model, in which $\hat{W}_\infty$ appears as a hidden current algebra. For the quantum $\hat{W}_\infty$ algebra at level $k=1$, an infinite set of commuting quantum charges in explicit and closed form was successfully constructed. Using them, a completely integrable quantum KP hierarchy is constructed in the Hamiltonian form. A two-boson realization of the quantum $\hat{W}_\infty$ currents has played a crucial role in this exploration.

I. INTRODUCTION

In the previous article,[1] we have extensively explored the interrelationship among the Kadomtsev–Petviashvili (KP) hierarchy,[2] nonlinear $\hat{W}_\infty$ algebra,[3] and conformal noncompact $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ coset model[4] at the classical level. We have shown that the KP hierarchy and $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ share a common $\hat{W}_\infty$ symmetry, which appears both as a (the second) Hamiltonian structure in the former[5,3] and a hidden current algebra in the latter.[6] Moreover, the well-known set of infinitely many involutive KP conserved charges give rise to an infinite $\hat{W}_\infty$ symmetry in the $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ model, which keeps the $\hat{W}_\infty$ algebra invariant. Because of the connection of the coset model to the black hole in 2D string theory,[7] it is necessary to study these issues at the quantum level. In particular, in this article we are interested in constructing a quantum version (or deformation) of the KP hierarchy which remains completely integrable.

As is well-known, the essence of the complete integrability, either at the classical or quantum level, is the existence of a complete set of conserved charges (many as basic variables) that are mutually commuting and independent of each other (involutive). This is known to be the case for the classical KP hierarchy.[5] The main difficulty in obtaining a quantum integrable KP hierarchy is to prove the existence of infinitely many involutive quantum charges.

To this end, it is natural to exploit the above-mentioned interrelationship among the KP hierarchy, $\hat{W}_\infty$ algebra, and $\mathrm{SL}(2,R)/\mathrm{U}(1)$ model. Working in the Hamiltonian formalism, quantization of the KP hierarchy is reduced to deforming its classical Hamiltonian structure and Hamiltonian functions. We choose to quantize the second KP Hamiltonian structure (or $\hat{W}_\infty$) which, as we have shown in Ref. 1, appears naturally in the $\mathrm{SL}(2,R)/\mathrm{U}(1)$ model. Therefore a consistent quantum deformation of $\hat{W}_\infty$ may be achieved by quantizing the coset model. On the other hand, to deform the KP Hamiltonian functions is a much harder problem, because there are no general rules for doing this for highly nonlinear Hamiltonians.

To be more concrete, let us recall in brief some basic facts about the classical KP hierarchy. It is an infinite set of evolution equations in various times t_m $(m=1,2,...,)$ in the Lax form

$$\frac{\partial L}{\partial t_m}=[(L^m)_+,L], \tag{1.1}$$

0022-2488/93/34(12)/5872/25/$6.00

where L is the pseudodifferential operator

$$L=D+\sum_{r=0}^{\infty} u_r D^{-r-1},\ D\equiv\partial/\partial z, \tag{1.2}$$

with coefficients u_r being functions of z and t_m and $(L^m)_+$ denoting the differential part of L^m. A fundamental property of hierarchy (1.1) is that it is Hamiltonian, namely, it can be put into the form

$$\frac{\partial u_r(z)}{\partial t_m}=\left\{u_r(z),\ \oint_0 H_{m+1}(w)dw\right\}. \tag{1.3}$$

Here the Hamiltonian functions are given by

$$H_{m+1}=\frac{1}{m}\,\mathrm{Res}\ L^m, \tag{1.4}$$

with Res L^m standing for the coefficient of the D^{-1} term in L^m. And the brackets are the Poisson ones

$$\{u_r(z),u_s(w)\}=k_{rs}(z)\delta(z-w), \tag{1.5}$$

with the two choices of k_{rs} explicitly given in Refs. 3 or 1, which, respectively, define the first[8] and second[5] KP Hamiltonian structures. In this article, we will focus only on the second KP Hamiltonian structure, the nonlinear classical $\hat{W}_\infty$ algebra, from which the first Hamiltonian structure $W_{1+\infty}$ (Refs. 8–10) can be obtained by appropriate contraction.[5,1]

Equations (1.3)–(1.5) imply that the infinite number of independent charges, $Q_m\equiv\oint H_m(z)dz$, are both conserved and in involution

$$\{Q_n,Q_{m+1}\}=\frac{\partial Q_n}{\partial t_m}=0. \tag{1.6}$$

To obtain the quantum KP hierarchy, we need to seek a quantum deformation of the $\hat{W}_\infty$ algebra (1.5). Unlike the case of (linear) Lie algebras, quantizing the nonlinear algebra $\hat{W}_\infty$ is quite a nontrivial task: Simply changing the Poisson brackets to commutators does not ensure closure of the algebra. Fortunately, we have at hand an elegant free field realization of the classical $\hat{W}_\infty$, which naturally appears in the $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ model.[6,1] Hence quantization of the coset model in the free field description[11] is expected to yield a consistent quantum $\hat{W}_\infty$ algebra in terms of the currents in the model. As we will see, the quantum $\hat{W}_\infty$ obtained in this manner receives intriguing deformations. The current–current commutators acquire both nonvanishing central terms and additional linear and nonlinear terms, and furthermore these terms in general depend on the level k, an essential parameter for the quantized coset model. This makes it very difficult to construct the desired[12,6] set of infinitely many commuting quantum $\hat{W}_\infty$ charges Q_m, which may be used to generate the quantum KP hierarchy. However, as reported in Ref. 13, such quantum $\hat{W}_\infty$ charges do exist, at least, at the level $k=1$. In this article we will give a rigorous proof for their explicit construction.

This article is organized as follows. In Sec. II, we generate the quantum $\hat{W}_\infty$ currents, by means of the operator product expansion (OPE), from the quantized $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ currents, with the help of their free boson realizations. These currents are shown to form a closed algebra, which is a quantum deformation of $\hat{W}_\infty$ and denoted as $\hat{W}_\infty(k)$. Some very useful OPE's for the quantum $\hat{W}_\infty$ currents are derived. Section III is then devoted to the explicit construction of an infinite number of commuting quantum $\hat{W}_\infty$ charges in the case with level

$k=1$, by means of conformal field theory techniques. In Sec. IV, we use these charges to generate a set of infinitely many compatible quantum flows, which are then justified to be a quantum deformation of the KP hierarchy and have a natural free field realization in terms of two bosons. [We make the following remark for caution. The KP flows we are going to quantize are flows in the space of functions $u_r(z)$; $r=0,1,2,\ldots$, of one variable z. Their quantization is principally different from the quantization of the original KP equation considered as a flow in the space of a function of two variables.]

II. QUANTUM $\hat{W}_\infty$ AND ITS TWO-BOSON REALIZATION

Usually at the quantum level a (linear) Lie algebra may acquire a central extension, due to normal ordering of operators. However, for a nonlinear algebra, the emergence of central terms alone in quantum corrections would violate closure, unless they are accompanied by additional (mostly nonlinear) terms. There are no general rules for writing down the additional terms. So quantization of a nonlinear algebra has to be done case by case. A free field realization, if available, would be very helpful.

Fortunately for the classical $\hat{W}_\infty$, we have previously established[6,1] a free field realization in terms of two bosons, which in the KP basis reads

$$L=D+\sum_{r=0}^{\infty} u_r D^{-r-1}=D+\bar{j}\,\frac{1}{D-(\bar{j}+j)}\,j, \tag{2.1}$$

with $\bar{j}(z)=\bar{\phi}'(z)$, $j(z)=\phi'(z)$, the free boson currents. Moreover, this realization appears naturally in the classical conformal $\mathrm{SL}(2,R)/\mathrm{U}(1)$ coset model through the following product expansion:

$$\psi_+(z)\psi_-(z')=\sum_{r=0}^{\infty} u_r(z)\,\frac{(z-z')^r}{r!}, \tag{2.2}$$

with the parafermionic currents ψ_+, ψ_- given by

$$\psi_+=\bar{j}e^{\bar{\phi}+\phi},\quad \psi_-=je^{-\bar{\phi}-\phi} \tag{2.3}$$

in terms of the two free bosons $\bar{\phi}$ and ϕ. (Here classically, without loss of generality, level k is set to 1.) The main observation to our success in quantizing $\hat{W}_\infty$ is that a consistent deformation for the $\hat{W}_\infty$ generators that necessarily leads to a closed algebra at the quantum level should naturally follow from quantizing the conformal coset model. (This observation was inspired by a work of Bakas and Kiritsis.[12])

Thus, let us start with the bosonized quantum $\mathrm{SL}(2,R)_k$ currents[11]

$$J_\pm=\sqrt{\frac{k}{2}}\,e^{\pm\sqrt{2/k}\phi_3}\left(\phi_1'\mp i\sqrt{1-\frac{2}{k}}\,\phi_2'\right)e^{\pm\sqrt{2/k}\phi_1},\quad J_3=-\sqrt{\frac{k}{2}}\,\phi_3', \tag{2.4}$$

where ϕ_i $(i=1,2,3)$ denote three free bosons, and k is the level parameter of the quantized model. Also we have set the Planck constant $\hbar=1$. Otherwise, k should be replaced by $k/\hbar$. [Note that each quantum current ϕ_i $(i=1,2,3)$ has the dimension $\hbar^{1/2}$.] Hence, the parameter $p\equiv k^{-1}$ plays the role of the Planck constant as the essential parameter in quantum corrections (or quantum deformation). In Eq. (2.4), we have included necessary quantum corrections in the currents. The classical limit is recovered by first rescaling $\phi_i\to\sqrt{k}\phi_i$, $J_i\to kJ_i$ and then letting $k\to\infty$.[1] Now by gauging away the U(1) current J_3 or simply setting $\phi_3=0$, from Eq. (2.7) we obtain the following parafermonic currents of the quantized $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ model:[14]

$$\psi_+(z;p)=\tfrac{1}{2}[(1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j]e^{\sqrt{p}(\bar{\phi}+\phi)},$$
$$\psi_-(z;p)=\tfrac{1}{2}[(1-\sqrt{1-2p})\bar{j}+(1+\sqrt{1-2p})j]e^{-\sqrt{p}(\bar{\phi}+\phi)}. \tag{2.5}$$

Here $\psi_\pm(z;p)\equiv J_\pm\sqrt{p}$, and we have written the two bosons $\phi_1(z)$ and $\phi_2(z)$, of the same signature, satisfying

$$\phi_i(z)\phi_j(z')\sim\delta_{ij}\log(z-z'),\quad i,j=1,2 \tag{2.6}$$

as a pair of complex bosons $\phi(z)=(1/\sqrt{2})(\phi_1-i\phi_2)$, $\bar{\phi}(z)=(1/\sqrt{2})(\phi_1+i\phi_2)$. Correspondingly, their currents $\bar{j}(z)$, $j(z)$ satisfy the standard OPE's

$$\bar{j}(z)j(z')\sim\frac{1}{(z-z')^2},\quad \bar{j}(z)\bar{j}(z')\sim j(z)j(z')\sim 0. \tag{2.7}$$

Generalizing the classical equation (2.2), we use the whole OPE (up to all orders), $\psi_+(z)\psi_-(z')$, of the $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ currents (2.5)

$$\psi_+(z;p)\psi_-(z';p)=\epsilon^{-2p}\left|\epsilon^{-2}+\sum_{r=0}^{\infty}u_r(z;p)\frac{\epsilon^r}{r!}\right| \tag{2.8}$$

(with $\epsilon\equiv z-z'$) to generate the quantum $\hat{W}_\infty(p)$ generators $u_r(z;p)$ in the KP basis. A closed expression for all $u_r(z;p)$ can be derived as follows.

Proposition 1: The expansion coefficients $u_r(p,z)$ in the OPE (2.8) are given by

$$\begin{aligned}
u_r(z;p)=&\frac{1}{4}\sum_{r_{k+1}=0}^{r}\ \sum_{r_k=r_{k+1}}^{r}\ \sum_{r_{k-1}=r_k}^{r}\cdots\\
&\times\sum_{r_1=r_2}^{r}\frac{(-1)^{r-r_k+r_{k+1}}\sqrt{p}^{r_k-r_{k+1}}r!}{r_{k+1}!(r_k-r_{k+1})!(r_{k-1}-r_k+1)!\cdots(r_1-r_2+1)!(r-r_1+1)!}\\
&\times((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)((1-\sqrt{1-2p})\bar{j}+(1+\sqrt{1-2p})j)^{(r_{k+1})}\\
&\times(\bar{j}+j)^{(r_{k-1}-r_k)}\cdots(\bar{j}+j)^{(r_1-r_2)}(\bar{j}+j)^{(r-r_1)}+\frac{(-1)^{r+1}\sqrt{p}}{(r+2)(r+1)}\\
&\times(\bar{j}+j)^{(r+1)}+\frac{(-1)^r\sqrt{p}}{2(r+1)}((1-\sqrt{1-2p})\bar{j}+(1+\sqrt{1-2p})j)^{(r+1)}\\
&+\sum_{r_{k+2}=0}^{r}\ \sum_{r_{k+1}=r_{k+2}}^{r}\cdots\sum_{r_1=r_2}^{r}\frac{(-1)^{r-r_{k+2}}\sqrt{p}^{r_{k+2}+2}r!}{(r_{k+2}+2)!(r_{k+1}-r_{k+2}+1)!\cdots(r_1-r_2+1)!(r-r_1+1)!}\\
&\times(\bar{j}+j)^{(r_{k+1}-r_{k+2})}\cdots(\bar{j}+j)^{(r_1-r_2)}(\bar{j}+j)^{(r-r_1)}\\
&+\frac{1}{2}\sum_{r_{k+1}=0}^{r}\ \sum_{r_k=r_{k+1}}^{r}\cdots\sum_{r_1=r_2}^{r}\frac{(-1)^{r-r_{k+1}+1}\sqrt{p}^{r_{k+1}+2}r!}{(r_{k+1}+1)!(r_k-r_{k+1}+1)!\cdots(r_1-r_2+1)!(r-r_1+1)!}\\
&\times((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)(\bar{j}+j)^{(r_k-r_{k+1})}\cdots(\bar{j}+j)^{(r_1-r_2)}(\bar{j}+j)^{(r-r_1)}
\end{aligned}$$

$$+\frac{1}{2}\sum_{r_{k+2}=0}^{r}\sum_{r_{k+1}=r_{k+2}}^{r}\cdots\sum_{r_1=r_2}^{r}\frac{(-1)^{r-r_{k+1}+r_{k+2}+1}\sqrt{p}^{r_{k+1}-r_{k+2}+2}r!}{r_{k+2}!(r_{k+1}-r_{k+2}+1)!\cdots(r_1-r_2+1)!(r-r_1+1)!}$$

$$\times((1-\sqrt{1-2p})\bar{j}+(1+\sqrt{1-2p})j)^{(r_{k+2})}$$

$$\times(\bar{j}+j)^{(r_k-r_{k+1})}\cdots(\bar{j}+j)^{(r_1-r_2)}(\bar{j}+j)^{(r-r_1)}, \tag{2.9}$$

where $j^{(r)}\equiv(\partial_z)^r j$, etc.

We observe that the quantum $\hat{W}_\infty(p)$ generators (2.9) depend only on the currents $\bar{j}$ and j, as in the classical case. The normal ordering on the current operators on the right side of Eq. (2.9) is understood. But we suppress the notation for convenience. In case there is confusion, we will put a *dot* symbol between two operators to denote their standard OPE to all orders.

Lemma 1:

$$e^{\sqrt{p}(\bar{\phi}+\phi)}(z)\cdot e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)=\epsilon^{-2p}e^{\sqrt{p}(\bar{\phi}+\phi)}(z)e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon), \tag{2.10}$$

$$\bar{j}(z)\cdot e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)=(\bar{j}-\epsilon^{-1}\sqrt{p})(z)e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon), \tag{2.11}$$

$$e^{\sqrt{p}(\bar{\phi}+\phi)}(z)\cdot\bar{j}(z-\epsilon)=e^{\sqrt{p}(\bar{\phi}+\phi)}(z)(\bar{j}-\epsilon^{-1}\sqrt{p})(z-\epsilon) \tag{2.12}$$

and Eqs. (2.14) and (2.15) hold for the $\bar{j}\leftrightarrow j$ interchange.

Proof: We show, e.g., Eq. (2.11) by a straightforward calculation

$$\bar{j}(z)\cdot e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)=\bar{j}(z)\cdot\sum_{m=0}^{\infty}\frac{(-\sqrt{p})^m}{m!}(\bar{\phi}+\phi)^m(z-\epsilon)$$

$$=\bar{j}(z)\sum_{m=0}^{\infty}\frac{(-\sqrt{p})^m}{m!}(\bar{\phi}+\phi)^m(z-\epsilon)+\sum_{m=1}^{\infty}\frac{(-\sqrt{p})^m}{(m-1)!\epsilon}(\bar{\phi}+\phi)^{m-1}$$

$$\times(z-\epsilon)$$

$$=\bar{j}(z)e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)-\epsilon^{-1}\sqrt{p}e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon),$$

where we have used Eq. (2.6). Q.E.D.

Now the proof of Proposition 1: From Lemma 1 we have, for example,

$$\bar{j}e^{\sqrt{p}(\bar{\phi}+\phi)}(z)\cdot je^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)=\epsilon^{-2p}[\bar{j}e^{\sqrt{p}(\bar{\phi}+\phi)}(z)je^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)+\epsilon^{-2}(1+p)e^{\sqrt{p}(\bar{\phi}+\phi)}$$

$$\times(z)e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)-\epsilon^{-1}\sqrt{p}\bar{j}e^{\sqrt{p}(\bar{\phi}+\phi)}(z)e^{-\sqrt{p}(\bar{\phi}+\phi)}$$

$$\times(z-\epsilon)-\epsilon^{-1}\sqrt{p}e^{\sqrt{p}(\bar{\phi}+\phi)}(z)je^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)]. \tag{2.13}$$

The operator product of two parafermion currents

$$I(p)\equiv\psi_+(z;p)\cdot\psi_-(z-\epsilon;p)$$

$$=\tfrac{1}{4}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar{\phi}+\phi)}(z)\cdot((1-\sqrt{1-2p})\bar{j}$$

$$+(1+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon) \tag{2.14}$$

then becomes, after some reorganizations

$$I(p)=\tfrac{1}{4}\epsilon^{-2p}[((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar{\phi}+\phi)}(z)\times((1-\sqrt{1-2p})\bar{j}$$
$$+(1+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)-2\epsilon^{-1}\sqrt{p}((1+\sqrt{1-2p})\bar{j}$$
$$+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar{\phi}+\phi)}(z)e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)-2\epsilon^{-1}\sqrt{p}e^{\sqrt{p}(\bar{\phi}+\phi)}(z)((1-\sqrt{1-2p})\bar{j}$$
$$+(1+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)+4\epsilon^{-2}e^{\sqrt{p}(\bar{\phi}+\phi)}(z)e^{-\sqrt{p}(\bar{\phi}+\phi)}(z-\epsilon)]. \tag{2.15}$$

By expanding Eq. (2.15) in powers of ϵ, and separating the terms with powers of ϵ^{-2p-2} and ϵ^{-2p-1} from the rest, it follows that

$$I(p)=\epsilon^{-2p}\Bigg[\epsilon^{-2}+\tfrac{1}{4}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)\sum_{k=0}^{\infty}\frac{\sqrt{p}^k}{k!}\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)!}\right.$$
$$\left.\times(\bar{j}+j)^{(m)}\epsilon^{m+1}\right)^k\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}((1-\sqrt{1-2p})\bar{j}^{(n)}+(1+\sqrt{1-2p})j^{(n)})\epsilon^n$$
$$-\sqrt{p}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+2)!}(\bar{j}+j)^{(m+1)}\epsilon^m+\tfrac{1}{2}\sqrt{p}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}((1-\sqrt{1-2p})\bar{j}^{(n+1)}$$
$$+(1+\sqrt{1-2p})j^{(n+1)})\epsilon^n+\sum_{k=0}^{\infty}\frac{\sqrt{p}^{k+2}}{(k+2)!}\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)!}(\bar{j}+j)^{(m)}\epsilon^m\right)^{k+2}\epsilon^k$$
$$-\tfrac{1}{2}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)\sum_{k=0}^{\infty}\frac{\sqrt{p}^{k+2}}{(k+1)!}\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)!}\right.$$
$$\left.\times(\bar{j}+j)^{(m)}\epsilon^m\right)^{k+1}\epsilon^k-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}((1-\sqrt{1-2p})\bar{j}^{(n)}+(1+\sqrt{1-2p})j^{(n)})\epsilon^n$$
$$\times\sum_{k=0}^{\infty}\frac{\sqrt{p}^{k+2}}{(k+1)!}\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)!}(\bar{j}+j)^{(m)}\epsilon^m\right)^{k+1}\epsilon^k\Bigg]. \tag{2.16}$$

We observe that the coefficient of the ϵ^{-2p-2} term is unity and all ϵ^{-2p-1} terms cancel against each other; thus we have verified the powers in expression (2.8). Now we change the ordering of multisummations to collect terms with the same power in ϵ together. Then one can easily read off Eq. (2.9) from the resulting expression of $I(p)$. Q.E.D.

Let us introduce the quantum KP operator as

$$L(z;p)=D+\sum_{r=0}^{\infty}u_r(z;p)D^{-r-1}, \tag{2.17}$$

with $u_r(z;p)$ precisely given by Eq. (2.9). It was proven in our previous article[1] that the bilocal classical function $\bar{j}e^{\bar{\phi}+\phi}(z)je^{-\bar{\phi}-\phi}(z-\epsilon)$ corresponds to the pseudodifferential operator $\bar{j}[1/D-(\bar{j}+j)]j$. Here we define a correspondence between the quantum KP operator (2.17) and the OPE (2.8) parallel to the classical case: The bilocal operator $F(z,z')=\Sigma_r p_r(z)(z-z')^r/r!$ is said to correspond to a pseudodifferential operator $P(z)=\Sigma_r p_r(z)D^{-r-1}$ of the same set of coefficient functions, and is denoted as $F(z,z')\Leftrightarrow P(z)$. The first two terms of Eq. (2.16) can be reformulated as

$$\epsilon^{-2}+\tfrac{1}{4}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar\phi+\phi)}(z)((1-\sqrt{1-2p})\bar{j}$$
$$+(1+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar\phi+\phi)}(z-\epsilon)$$
$$\Leftrightarrow D+\tfrac{1}{4}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)\frac{1}{D-\sqrt{p}(\bar{j}+j)}$$
$$\times((1-\sqrt{1-2p})\bar{j}+(1+\sqrt{1-2p})j), \tag{2.18}$$

where we have set $\epsilon^{-2}\Leftrightarrow D$. It is obvious that Eq. (2.18) is a simple deformation of Eq. (2.1). The remaining terms in Eq. (2.16) represent additional quantum corrections.

The complete structure of the quantum $\hat{W}_\infty(p)$ algebra can be manifested by the OPE's between two currents $u_r(z;p)$ and $u_s(w;p)$, as usual in conformal field theory. In principle, these OPE's can be extracted from the following OPE of four $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ parafermionic currents:

$$L(z;p)\,L(w;p)\Leftrightarrow(\epsilon\sigma)^{2p}(\psi_+(z;p)\psi_-(z-\epsilon;p))(\psi_+(w;p)\psi_-(w-\sigma;p)). \tag{2.19}$$

The closure of the quantum $\hat{W}_\infty(p)$ algebra is ensured by the closure of the OPE's associated with the enveloping algebra of the $\mathrm{SL}(2,R)_k$ currents (2.4) in the neutral sector in the conformal model. In fact, in the $\mathrm{SL}(2,R)_k$ neutral sector, currents are always combinations of the products of $J_+J_-^{(n)}$, J_3 and their derivatives. Imposing the $J_3=0$ constraint selects combinations of the products of $\psi_+\psi_-^{(n)}$ and their derivatives only. That they are closed within the $\hat{W}_\infty(p)$ currents $u_r(z;p)$ in accordance to Eq. (2.8) is thus guaranteed by the closure of the enveloping $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ algebra in the neutral sector. Therefore we have achieved a consistent quantum deformation of $\hat{W}_\infty$, with the help of the free field realization of the $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ coset model. However, despite the crucial role of the model in the construction, once we write the algebra in the form of the OPE's among u_r and u_s, the currents u_r may be considered as independent of each other, and the associativity of the OPE's automatically leads to closed Jacobi identities, which do not require the use of the two-boson representation (2.9). What we have obtained is thus justified to be a quantum version of the full $\hat{W}_\infty$ algebra with independent currents, which holds even beyond the context of the conformal coset model.

For the purpose of illustration and later use, let us give some explicit expressions for the quantum $\hat{W}_\infty(p)$. From Eq. (2.9), the first few quantum $\hat{W}_\infty(p)$ generators read

$$u_0(p)=(1-2p)\bar{j}j-\tfrac{1}{2}\sqrt{1-2p}\,\sqrt{p}(\bar{j}'-j'),$$
$$u_1(p)=-\tfrac{1}{2}((1-p)\sqrt{1-2p}+1-2p)\bar{j}j'+\tfrac{1}{2}((1-p)\sqrt{1-2p}-1+2p)\bar{j}'j-\tfrac{1}{2}p\sqrt{1-2p}(\bar{j}\bar{j}'-jj')$$
$$+\tfrac{1}{12}(3\sqrt{1-2p}-1)\sqrt{p}\bar{j}''-\tfrac{1}{12}(3\sqrt{1-2p}+1)\sqrt{p}j''+(1-\tfrac{3}{2}p)\sqrt{p}(\bar{j}j^2+\bar{j}^2j)$$
$$+\tfrac{1}{6}p\sqrt{p}(\bar{j}^3+j^3), \tag{2.20}$$
$$u_2(p)=\tfrac{1}{2}((1-p)\sqrt{1-2p}+1-2p)\bar{j}j''-\tfrac{1}{2}((1-p)\sqrt{1-2p}-1+2p)\bar{j}''j-\tfrac{1}{2}p\bar{j}'j'+\tfrac{1}{4}(2\sqrt{1-2p}$$
$$-1)p\bar{j}'^2-\tfrac{1}{4}(2\sqrt{1-2p}+1)pj'^2+\tfrac{1}{2}p\sqrt{1-2p}(\bar{j}\bar{j}''-jj'')-\tfrac{1}{12}(2\sqrt{1-2p}-1)\sqrt{p}\bar{j}'''$$
$$+\tfrac{1}{12}(2\sqrt{1-2p}+1)\sqrt{p}j'''-((1-p)\sqrt{1-2p}+2-3p)\sqrt{p}\bar{j}jj'+((1-p)\sqrt{1-2p}-2$$
$$+3p)\sqrt{p}\bar{j}\bar{j}'j+((1-\tfrac{1}{2}p)\sqrt{1-2p}-1+\tfrac{3}{2}p)\sqrt{p}\bar{j}'j^2-((1-\tfrac{1}{2}p)\sqrt{1-2p}+1-\tfrac{3}{2}p)\sqrt{p}\bar{j}^2j'$$

$$-\tfrac{1}{2}(\sqrt{1-2p}+1)p\sqrt{p}\bar{j}^{\,2}\bar{j}'+\tfrac{1}{2}(\sqrt{1-2p}-1)p\sqrt{p}j^2j'+(1-p)p(\bar{j}j^3+\bar{j}^{\,3}j)$$
$$+(2-\tfrac{5}{2}p)p\bar{j}^{\,2}j^2+\tfrac{1}{4}p^2(\bar{j}^4+j^4).$$

We have checked that the quantum generators $W_i(k)$ $(k=p^{-1})$ constructed in Ref. 12 can be obtained under the following basis transformation:

$$W_2(k)=\frac{1}{(1-2p)}\,u_0(p),\quad W_3(k)=-4u_1(p)-2u_0'(p),$$
$$W_4(k)=16u_2(p)+16u_1'(p)+\frac{16}{5}\,u_0''(p)-\frac{16(6+5p)}{(16-17p)}\,(u_0u_0)(p), \tag{2.21}$$

where the local product (u_0u_0) is given by

$$(u_0u_0)(p,z)=\frac{1}{2}\,(1-2p)^2(\bar{j}j''+\bar{j}''j)-\frac{p}{2}\,(1-2p)\bar{j}'j'+\frac{p}{4}\,(1-2p)(\bar{j}'^2+j'^2)-\frac{1}{6}\,(1-2p)$$
$$\times\sqrt{1-2p}\,\sqrt{p}(\bar{j}'''-j''')-(1-2p)\sqrt{1-2p}\,\sqrt{p}\bar{j}j(\bar{j}'-j')+(1-2p)^2\bar{j}^{\,2}j^2. \tag{2.22}$$

Using these expressions, it is straightforward to calculate the first few OPE's for the quantum $\hat{W}_\infty(p)$, from the OPE (2.7) for the free boson currents. They read

$$u_0(z)u_0(w)=(1-2p)\left(\frac{2u_0(w)}{(z-w)^2}+\frac{u_0'(w)}{(z-w)}+\frac{(1+p)}{(z-w)^4}\right)+O(z-w),$$
$$u_0(z)u_1(w)=(1-2p)\left(\frac{3u_1(w)}{(z-w)^2}+\frac{u_1'(w)}{(z-w)}-\frac{2u_0(w)}{(z-w)^3}-\frac{2(1+p)}{(z-w)^5}\right)+O(z-w),$$
$$u_0(z)u_2(w)=(1-2p)\left(\frac{4u_2(w)}{(z-w)^2}+\frac{u_2'(w)}{(z-w)}-\frac{6u_1(w)}{(z-w)^3}+\frac{2(3+p)u_0(w)}{(z-w)^4}+\frac{6(1+p)}{(z-w)^6}\right)+O(z-w), \tag{2.23}$$
$$u_1(z)u_1(w)=\left(1-\frac{3p}{2}\right)\left(\frac{4u_2(w)}{(z-w)^2}+\frac{2u_2'(w)}{(z-w)}\right)+(1-p)\left(\frac{2u_1'(w)}{(z-w)^2}+\frac{u_1''(w)}{(z-w)}\right)+\frac{8(1-p)pu_0(w)}{(z-w)^4}$$
$$+\frac{4(1-p)pu_0'(w)}{(z-w)^3}+\frac{pu_0''(w)}{(z-w)^2}+\frac{(1+2p)pu_0'''(w)}{6(z-w)}+\frac{2p(u_0u_0)(w)}{(z-w)^2}+\frac{p(u_0u_0)'(w)}{(z-w)}$$
$$-\frac{4(1+p)(3-8p+2p^2)}{3(z-z')^6}.$$

From Eq. (2.23) we see that the quantum $\hat{W}_\infty(p)$ algebra is really p dependent. Its p-independent classical limit—$\hat{W}_\infty$—is recovered by taking $p\to 0$ after rescaling $u_r\to u_r/p$ and $[,]\to p\{,\}$. This is equivalent to the standard $\hbar\to 0$ limit (after setting $p\to p\hbar$). In this sense the $p=1$ case, corresponding to the $\mathrm{SL}(2,R)_k/\mathrm{U}(1)$ coset model with level $k=1$, may be interpreted as the "typical" quantum case. For this value of p, a lot of expressions simplify and it becomes possible to extract the most relevant $(z-w)^{-1}$ terms in the OPE's between $u_0(z)$ or $u_1(z)$ and arbitrary $u_s(z)$. In next section we will see that these OPE's provide sufficient information for constructing an infinite set of commuting quantum $\hat{W}_\infty$ charges. [Note that to obtain charge commutators from the OPE's, one needs a double integrations over z and w, so the terms in the OPE with other powers of $(z-w)^{-1}$ have no contribution to the commutators.]

Proposition 2: In the case of $p=1$, we have

$$u_0(z)u_s(w)=\frac{-1}{z-w}\,u_s'(w)+\text{terms in other powers of }(z-w)^{-1},$$

$$u_1(z)u_s(w)=\frac{-2}{z-w}\left[\sum_{l=1}^{s}(-1)^l\binom{s}{l}(u_0^{(l)}u_{s-l})+\frac{u_{s+1}'}{(s+1)}+\frac{(-1)^s u_0^{(s+2)}}{(s+1)(s+2)}\right](w) \tag{2.24}$$

$$+\text{terms in other powers of }(z-w)^{-1}.$$

Proof: Instead of using the explicit expression (2.9) of $u_s(z;p)$, we begin with the closed form (2.15). For the OPE $u_0(z)u_s(w)$ we can in fact obtain a general expression for arbitrary p. Note that the total derivatives in $u_0(z;p)$ [see Eq. (2.20)] will not contribute to the $(z-w)^{-1}$ terms in the OPE, so we have

$$\begin{aligned}J_1&\equiv u_0(z;p)\cdot\left[\epsilon^{-2}+\sum_{s=0}^{\infty}u_s(w;p)\,\frac{\epsilon^s}{s!}\right]\\
&=(1-2p)\bar{j}j(z)\cdot[\tfrac{1}{4}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar{\phi}+\phi)}(w)((1-\sqrt{1-2p})\bar{j}+(1\\
&\quad+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)-\tfrac{1}{2}\epsilon^{-1}\sqrt{p}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar{\phi}+\phi)}(w)\\
&\quad\times e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)-\tfrac{1}{2}\epsilon^{-1}\sqrt{p}e^{\sqrt{p}(\bar{\phi}+\phi)}(w)((1-\sqrt{1-2p})\bar{j}+(1+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar{\phi}+\phi)}\\
&\quad\times(w-\epsilon)+\epsilon^{-2}e^{\sqrt{p}(\bar{\phi}+\phi)}(w)e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)]+\text{terms in other powers of }(z-w)^{-1}.\end{aligned} \tag{2.25}$$

We extract from Eq. (2.25) the $(z-w)^{-1}$ terms and then reorganize them into the fashion of Eq. (2.15). In this way, one obtains

$$\begin{aligned}J_1&=\frac{1}{z-w}(1-2p)\partial_w[\tfrac{1}{4}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar{\phi}+\phi)}(w)((1-\sqrt{1-2p})\bar{j}+(1\\
&\quad+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)-\tfrac{1}{2}\epsilon^{-1}\sqrt{p}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar{\phi}+\phi)}(w)\\
&\quad\times e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)-\tfrac{1}{2}\epsilon^{-1}\sqrt{p}e^{\sqrt{p}(\bar{\phi}+\phi)}(w)((1-\sqrt{1-2p})\bar{j}+(1+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar{\phi}+\phi)}\\
&\quad\times(w-\epsilon)+\epsilon^{-2}e^{\sqrt{p}(\bar{\phi}+\phi)}(w)e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)]+\text{terms in other powers of }\frac{1}{z-w}\\
&=\frac{1}{z-w}\sum_{s=0}^{\infty}(1-2p)u_s'(w;p)\,\frac{\epsilon^s}{s!}+\text{terms in other powers of }(z-w)^{-1}.\end{aligned} \tag{2.26}$$

Comparing Eq. (2.26) with Eq. (2.25), we obtain the first equation of (2.24).

The same scheme works for the proof of the second equation of (2.24). According to Eq. (2.20), when $p=1$, u_1 reads

$$u_1(z)=-\tfrac{1}{2}\bar{j}j(\bar{j}+j)+\tfrac{1}{6}(\bar{j}^3+j^3)+\text{total derivatives}. \tag{2.27}$$

It is sufficient to address

$$\begin{aligned}J_2&\equiv u_1(z)\cdot\left[\epsilon^{-2}+\sum_{s=0}^{\infty}u_s(w)\frac{\epsilon^s}{s!}\right]\\
&=(-\tfrac{1}{2}\bar{j}j(\bar{j}+j)+\tfrac{1}{6}(\bar{j}^3+j^3))(z)\cdot[\tfrac{1}{4}((1+\sqrt{1-2p})\bar{j}+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar{\phi}+\phi)}(w)\\
&\quad\times((1-\sqrt{1-2p})\bar{j}+(1+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)-\tfrac{1}{2}\epsilon^{-1}\sqrt{p}((1+\sqrt{1-2p})\bar{j}\\
&\quad+(1-\sqrt{1-2p})j)e^{\sqrt{p}(\bar{\phi}+\phi)}(w)e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)-\tfrac{1}{2}\epsilon^{-1}\sqrt{p}e^{\sqrt{p}(\bar{\phi}+\phi)}(w)((1-\sqrt{1-2p})\bar{j}\\
&\quad+(1+\sqrt{1-2p})j)e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)+\epsilon^{-2}e^{\sqrt{p}(\bar{\phi}+\phi)}(w)e^{-\sqrt{p}(\bar{\phi}+\phi)}(w-\epsilon)]\\
&\quad+\text{terms in other powers of }(z-w)^{-1}.\end{aligned} \tag{2.28}$$

It turns out that (we skip the tedious calculation)

$$\begin{aligned}J_2&=\frac{-2}{z-w}\sum_{s=0}^{\infty}\left[\sum_{l=1}^{s}(-1)^l\binom{s}{l}(u_0^{(l)}u_{s-l})+\frac{u'_{s+1}}{(s+1)}+\frac{(-1)^s u_0^{(s+2)}}{(s+1)(s+2)}\right](w)\frac{\epsilon^s}{s!}\\
&\quad+\text{terms in other powers of }(z-w)^{-1},\end{aligned} \tag{2.29}$$

which leads to the desired equation. Q.E.D.

III. INVOLUTIVE QUANTUM $\hat{W}_\infty$ CHARGES

The quantum $\hat{W}_\infty$ algebra obtained in last section is a quantum version (deformation) of the second Hamiltonian structure (1.5) of the classical KP hierarchy (1.3). We intend to use it as the Hamiltonian structure of the yet-to-be established quantum KP hierarchy. What more we need is an infinite set of commuting quantum $\hat{W}_\infty$ charges, whose densities are the quantum deformation of the classical Hamiltonian functions (1.4).

For generic p, it is not hard to construct the first few quantum Hamiltonians[6,12]

$$\begin{aligned}&H_2(z;p)=\frac{1}{(1-2p)}u_0(z;p),\quad H_3(z;p)=u_1(z;p)+\tfrac{1}{2}u_0'(z;p),\\
&H_4(z;p)=u_2(z;p)+u_1'(z;p)+\frac{(5+4p)}{15}u_0''(z;p)+\frac{p}{(1-2p)}(u_0u_0)(z;p).\end{aligned} \tag{3.1}$$

Their charges $Q_m(p)\equiv\oint_0 H_m(z;p)dz$ $(m=2,3,4)$ indeed mutually commute. In principle, such construction may successively continue to higher orders with rapidly increasing labor and effort. Fortunately, in the "typical" quantum case with $p=1$, we can exploit the two general OPE's (2.24) to explicitly construct an infinite number of independent, commuting quantum $\hat{W}_\infty$ charges Q_m.

To start, let us first review in brief some useful facts for the local product of several local operators in the conformal field theory. First the local product $(AB)(z)$ of two local operators $A(z)$ and $B(z)$ is the $(w-z)^0$ term in their OPE, namely,

$$(AB)(z)=\oint_z \frac{A(w)B(z)}{w-z}\,dw \tag{3.2}$$

in which the small contour of integration encircles z. The action of the z derivative still satisfies the usual Leibniz rule

$$\partial_z(AB)(z)=((\partial_z A)B)(z)+(A\partial_z B)(z). \tag{3.3}$$

This local product is noncommutative, i.e., $(AB)(z)\neq(BA)(z)$; however they differ from each other only by total derivatives

$$\oint_0 (AB-BA)(z)dz=0. \tag{3.4}$$

According to definition (3.2), the operator product of $C(z)$ with the local product $(AB)(w)$ is given by

$$C(z)(AB)(w)=((C(z)A(w))B(w))+(A(w)(C(z)B(w))). \tag{3.5}$$

So, the multiple local products are generally nonassociative, e.g., $(A(BC))-(A(BC))\neq 0$, but one has the relation

$$(A(BC))-(B(AC))=((AB)C)-((BA)C). \tag{3.6}$$

Finally we define the symmetric local product of N local operators to be the totally symmetrized sum of their multiple local products taken from the left

$$\langle A_1A_2\cdots A_N\rangle=\frac{1}{N!}\sum_{P\{i\}}(\cdots((A_{i_1}A_{i_2})A_{i_3})\cdots A_{i_N}), \tag{3.7}$$

where $P\{i\}$ denotes the summation over all possible permutations.

Now we proceed to construct an infinite set of involutive quantum $\hat{W}_\infty$ charges (with $p=1$). Let us assign a degree 1 to ∂_z and $r+2$ to u_r. We assume that the quantum charge-density $H_m(z)$ is homogeneous and of degree m, without loss of generality. Furthermore we assume that the leading term of $H_m(z)$ which is linear in the highest-spin current u_{m-2} has a nonvanishing coefficient, which can be normalized to unity. This ensures the mutual independence of these charge densities. Therefore the most general form for $H_m(z)$ is a linear combination of multiple local products of the currents $u_r(z)$ and their derivatives

$$H_m(z)=\sum_l \sum_{\{i,a\}} C^{a_1a_2\cdots a_l}_{i_1i_2\cdots i_l}(m)(\cdots(u^{(a_1)}_{u_1}u^{(a_2)}_{i_2})\cdots u^{(a_l)}_{i_l})(z),\quad m=2,3,..., \tag{3.8}$$

where l $(=1,2,...,[m/2])$ is the number of currents u_r in the product (the maximal value of l being the integral part of $m/2$), $\{i,a\}$ stands for the set of all possible indices i's and a's satisfying $i_1+i_2+\cdots+i_l+a_1+a_2+\cdots+a_l=m-2l$, and $C^{a_1a_2\cdots a_l}_{i_1i_2\cdots i_l}(m)$ are constant coefficients. We want to determine these coefficients so that the corresponding charges $Q_m\equiv\oint H_m(z)dz$ commute with each other

$$\left[\oint_0 H_n(z)dz,\ \oint_0 H_m(w)dw\right]=0. \tag{3.9}$$

Proposition 3: The commutativity with Q_2, i.e.,

$$\left[\oint_0 H_2(z)dz,\ \oint_0 H_m(w)dw\right]=0 \tag{3.10}$$

is always satisfied.

Proof: Recall that for arbitrary local operators $A(z)$ and $B(z)$, we can rewrite the commutator of their integrals in terms of their OPE

$$\left[\oint_0 A(z)dz,\ \oint_0 B(w)dw\right]=\oint_0 dz\ \oint_z dw\, A(z)\cdot B(w) \tag{3.11}$$

through a continuous deformation of the integration contour. Hence (note H_2 has to be u_0 according to the homogeneity assumption)

$$K\equiv\left[\oint_0 H_2(z)dz,\ \oint_0 H_m(w)dw\right]=\oint_0 dz\ \oint_z dw\, u_0(z)\cdot H_m(w).$$

With arbitrary coefficients C's in Eq. (3.8) we have, by applying Eq. (3.5)

$$K=\oint_0 dz\ \oint_z dw \sum_{k=1}^{l}\sum_{l}\sum_{\{i,a\}} C_{i_1i_2\cdots i_l}^{a_1a_2\cdots a_l}(m)(\cdots(u_{i_1}^{(a_1)}u_{i_2}^{(a_2)})\cdots(u_0(z)\cdot u_{i_k}^{(a_k)}(w))\cdots u_{i_l}^{(a_l)})(w).$$

It follows that, by using the first equation of (2.24) and Eq. (3.3)

$$\begin{aligned}K&=\oint_0 dz\ \oint_z \frac{dw}{w-z}\sum_{k=1}^{l}\sum_{l}\sum_{\{i,a\}} C_{i_1i_2\cdots i_l}^{a_1a_2\cdots a_l}(m)((\cdots(u_{i_1}^{(a_1)}u_{i_2}^{(a_2)})\cdots u_{i_k}^{(a_k+1)})\cdots u_{i_l}^{(a_l)})(w)\\&=\oint_0 dz\ \oint_z \frac{dw}{w-z}\partial_w\Big(\sum_{l}\sum_{\{i,a\}} C_{i_1i_2\cdots i_l}^{a_1a_2\cdots a_l}(m)(\cdots(u_{i_1}^{(a_1)}u_{i_2}^{(a_2)})\cdots u_{i_l}^{(a_l)})(w)\Big)\\&=0.\end{aligned} \tag{3.12}$$

(Q.E.D.)

Thus the first set of nontrivial equations in Eq. (3.9) start with $n=3$

$$\left[\oint_0 H_3(z)dz,\ \oint_0 H_m(w)dw\right]=0. \tag{3.13}$$

Here H_3 can only be u_1 plus a derivative of u_0; the latter does not contribute to the charge Q_3. In the following we will show that all the charge density $H_m(z)$ modulo total derivatives, and thus all charges Q_m, are completely determined by Eq. (3.13) alone. Both amusingly and amazingly, as we will see later, the so-determined charges Q_m automatically commute with each other.

A direct step-by-step construction, with extensive use of the second equation of (2.24) and Eqs. (3.3)–(3.6), gives *uniquely* the first seven charges from Eq. (3.13) as follows:

$$Q_2=\oint u_0(z)dz,$$

$$Q_3= \oint u_1(z)dz, \quad Q_4= \oint (u_2-u_0u_0)(z)dz,$$

$$Q_5= \oint (u_3-6u_0u_1)(z)dz, \tag{3.14}$$

$$Q_6= \oint (u_4-12u_0u_2-12u_1u_1+8(u_0u_0)u_0)(z)dz,$$

$$Q_7= \oint (u_5-20u_0u_3-60u_1u_2+60(u_0u_0)u_1+60(u_0u_1)u_0)(z)dz,$$

$$Q_8= \oint (u_6-30u_0u_4-12u_1u_3-90u_2u_2+180(u_0u_0)u_2+180(u_0u_2)u_0$$

$$+360(u_1u_0)u_1+360(u_1u_1)u_0-180((u_0u_0)u_0)u_0)(z)dz.$$

Our key observation is that these charges Q_m share the following nice features:

(i) No term contains any derivative of u_r at all, if all local products of the currents are chosen to start from the left.

(ii) For every Q_m, its coefficients conspire to result in totally symmetric multiple (local) products of the currents involved.

(iii) All the seven charges are given by

$$Q_m= \oint_0 \sum_l \sum_{\{i\}} C_{i_1i_2\cdots i_l}(m)\langle u_{i_1}u_{i_2}\cdots u_{i_l}\rangle(z)dz, \tag{3.15}$$

with the elegant expression for the coefficients

$$C_{i_1i_2\cdots i_l}(m)=\frac{(-1)^{l-1}(l-1)!(m-2)!}{d_1!d_2!\cdots d_k!i_1!i_2!\cdots i_l!}, \tag{3.16}$$

where for given number of currents in the product l, the summation is over all partitions $\{i_k\}$ of $m-2l$, satisfying $i_1+i_2+\cdots+i_l=m-2l$, and here the d's denote the degeneracies in the partition: $0\leqslant i_1=i_2=\cdots=i_{d_1}<i_{d_1+1}=\cdots=i_{d_1+d_2}<\cdots=i_{d_1+d_2+\cdots+d_k(=l)}$. Note the coefficient of the leading linear term u_{m-2} is unity as desired.

Now let us show that these are actually true for arbitrary Q_m.

Proposition 4: Equation (3.16) provides a unique solution of the form (3.15) to Eq. (3.13) for arbitrary m.

Before proceeding to the proof, we need, for technical preparation

Lemma 2:

$$\oint_0 \langle \underline{A}_0A_1\cdots A_i\cdots A_N\rangle(z)dz= \oint_0 \langle A_0A_1\cdots A_i\cdots A_N\rangle(z)dz, \tag{3.17}$$

where the *bar* under A_0 on the left side indicates that it does not join in the symmetrization

$$\langle \underline{A}_0A_1\cdots A_i\cdots A_N\rangle\equiv\frac{1}{N!}\{(\cdots((A_0A_1)A_2)\cdots A_N)+(\cdots((A_0A_2)A_1)\cdots A_N)$$

$$+\text{all other permutations among } 1,2,\ldots,N\}. \tag{3.18}$$

So the left side is symmetrized with respect to the N indices $(1,2,...,N)$, while the right side is symmetrized with respect to all $N+1$ indices by definition.

Proof: In fact, we have a stronger result

$$\begin{aligned}
&N(\cdots((A_0A_1)A_2)\cdots A_N)+A_1(\cdots((A_0A_2)A_3)\cdots A_N)+(A_1A_2)(\cdots((A_0A_3)A_4)\cdots A_N)+\cdots\\
&\quad+(\cdots((A_1A_2)A_3)\cdots A_{N-1})(A_0A_N)+\text{permutations among } 1,2,...,N\\
&\quad=\sum_{k=1}^{N}(\cdots((A_1A_2)A_3)\cdots A_0)A_k)\cdots A_N)+A_0(\cdots((A_1A_2)A_3)\cdots A_N)\\
&\qquad+(A_0A_{N-1})(\cdots((A_1A_2)A_3)\cdots A_{N-2})A_N)+\cdots+(\cdots((A_0A_2)A_3)\cdots A_{N-1})(A_1A_N)\\
&\qquad+\text{permutations among } 1,2,...,N.
\end{aligned}\tag{3.19}$$

After integration, it yields Eq. (3.17) by using Eq. (3.4).

Consider, for example, the case with $N=3$. For the right side of Eq. (3.19), with repeated use of Eq. (3.6), we have

$$\begin{aligned}
&((A_0A_1)A_2)A_3+((A_1A_0)A_2)A_3+((A_1A_2)A_0)A_3+A_0((A_1A_2)A_3)+(A_0A_2)(A_1A_3)+(1,2,3)\ \text{perm.}\\
&\quad=2((A_0A_1)A_2)A_3+(A_1(A_0A_2))A_3-(A_0(A_1A_2))A_3+((A_1A_2)A_0)A_3+A_0((A_1A_2)A_3)\\
&\qquad+(A_0A_2)(A_1A_3)+(1,2,3)\ \text{perm.}\\
&\quad=2((A_0A_1)A_2)A_3+((A_0A_2)A_1)A_3+A_1((A_0A_2)A_3)+(A_1A_2)(A_0A_3)+(1,2,3)\ \text{perm.},
\end{aligned}$$

which is just the left side of Eq. (3.19). The general validity is proven by induction. Q.E.D.

Now the proof of Proposition 4: From Eq. (3.11), we see Eq. (3.13) is equivalent to

$$\oint_0 dz\ \oint_z dw\, u_1(z)H_m(w)=0.$$

Applying the second equation of (2.24), and performing the w integration, we rewrite it as

$$\begin{aligned}
&\oint_0 \sum_l \sum_{\{i\}} C_{i_1i_2\cdots i_l}(m)\sum_{a=1}^{l}\left\langle u_{i_1}\cdots u_{i_{a-1}}\left[\sum_{k=1}^{i_a}(-1)^k\binom{i_a}{k}(u_0^{(k)}u_{i_a-k})+\frac{u'_{i_a+1}}{(i_a+1)}\right.\right.\\
&\qquad\left.\left.+\frac{(-1)^{i_a}u_0^{(i_a+2)}}{(i_a+1)(i_a+2)}\right|u_{i_{a+1}}\cdots u_{i_l}\right\rangle(z)dz=0.
\end{aligned}\tag{3.20}$$

Note that each term has only one derivative on one of the currents. To verify Eq. (3.16), one needs to show that all the terms in Eq. (3.20) with the same order of derivative on one of the u_r's must cancel each other. In particular, collecting the terms with a first-order derivative we want to verify that

$$\oint_0 \sum_l \sum_{\{i\}} C_{i_1i_2\cdots i_l}(m)\sum_{a=1}^{l}\left\langle u_{i_1}\cdots u_{i_{a-1}}\left|-i_a(u'_0u_{i_{a-1}})+\frac{u'_{i_a+1}}{(i_a+1)}\right|u_{i_{a+1}}\cdots u_{i_l}\right\rangle(z)dz=0,\tag{3.21}$$

which can further be decomposed into

$$\oint_0 \sum_{\{i\}_{l+1}} C_{i_1^{d_1} i_2^{d_2} \cdots i_k^{d_k}}(m) \sum_{a=1}^{k} \frac{d_a}{i_a+1} \langle u_{i_1}^{d_1} \cdots u_{i_a}^{d_a-1} u'_{i_a+1} u_{i_{a+1}}^{d_{a+1}} \cdots u_{i_k}^{d_k} \rangle (z) dz$$
$$= \oint_0 \sum_{\{j\}_l} C_{j_1^{c_1} j_2^{c_2} \cdots j_n^{c_n}}(m) \sum_{a=1}^{n} c_a j_a \langle u_{j_1}^{c_1} \cdots (u'_0 u_{j_a-1}) u_{j_a}^{c_a-1} u_{j_{a+1}}^{c_{a+1}} \cdots u_{j_n}^{c_n} \rangle (z) dz \tag{3.22}$$

for each l. (For convenience, we will call l, the number of currents in a product, the level of the term. Do not confuse it with the level of the model.) Here the C's on the left side are level-$(l+1)$ coefficients and those on the right side are level-l ones: $\Sigma\, d_a = l+1$, $\Sigma\, d_a i_a = m - 2(l+1)$, and $\Sigma\, c_b = l$, $\Sigma\, c_a j_b = m - 2l$, with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_n$.

For terms on the left side having a derivative on the highest-spin current u_{i_k}, we integrate by parts and turn such terms into those containing no derivative on u_{i_k}. Terms of the latter property will be called irreducible; in the following we will assume each term in a given partition of $m-2l$ has been turned irreducible in this way. To handle the local product $(u'_0 u_{j_a-1})$ in the middle of the right side of Eq. (3.22), which is symmetrized as a whole with other currents, one can use Lemma 2 to symmetrize all the currents involved in a given partition. Let us prove Eq. (3.22) partition by partition.

We will do this by induction for both the level l and the first index i_1 in the partition. First, on the left side of Eq. (3.22) which is already symmetrized, there is a term (after integration by parts) with the first index $i_1 = 0$ and of the form $u'_0 u_0^{d_1-1} u_{i_2}^{d_2} \cdots u_{i_{k-1}}^{d_{k-1}} u_{i_k}^{d_k-1} u_{i_k+1}$, with coefficient

$$-\frac{d_1 d_k}{i_k+1} C_{0^{d_1} i_2^{d_2} \cdots i_k^{d_k}}(m). \tag{3.23}$$

Terms of the same partition on the right side appear as

$$C_{0^{d_1-2} 1 i_2^{d_2} \cdots i_{k-1}^{d_{k-1}} i_k^{d_k-1} (i_k+1)}(m) \langle (u'_0 u_0) u_0^{d_1-2} u_{i_2}^{d_2} \cdots u_{i_{k-1}}^{d_{k-1}} u_{i_k}^{d_k-1} u_{i_k+1} \rangle + (i_2+1)$$
$$\times C_{0^{d_1-1} i_2^{d_2-1} (i_2+1) i_3^{d_3} \cdots i_{k-1}^{d_{k-1}} i_k^{d_k-1} (i_k+1)}(m) \langle (u'_0 u_{i_2}) u_0^{d_1-1} u_{i_2}^{d_2-1} u_{i_3}^{d_3} \cdots u_{i_{k-1}}^{d_{k-1}} u_{i_k}^{d_k-1} u_{i_k+1} \rangle + \cdots$$
$$+ (i_{k-1}+1) C_{0^{d_1-1} i_2^{d_2} \cdots i_{k-1}^{d_{k-1}-1} (i_{k-1}+1) i_k^{d_k-1} (i_k+1)}(m)$$
$$\times \langle (u'_0 u_{i_{k-1}}) u_0^{d_1-1} u_{i_2}^{d_2} \cdots u_{i_{k-1}}^{d_{k-1}-1} u_{i_k}^{d_k-1} u_{i_k+1} \rangle + 2(i_k+1) C_{0^{d_1-1} i_2^{d_2} \cdots i_{k-1}^{d_{k-1}} i_k^{d_k-2} (i_k+1)^2}(m)$$
$$\times \langle (u'_0 u_{i_k}) u_0^{d_1-1} u_{i_2}^{d_2} \cdots u_{i_{k-1}}^{d_{k-1}} u_{i_k}^{d_k-2} u_{i_k+1} \rangle + (i_k+2) C_{0^{d_1-1} i_2^{d_2} \cdots i_{k-1}^{d_{k-1}} i_k^{d_k-1} (i_k+2)}(m)$$
$$\times \langle (u'_0 u_{i_k+1}) u_0^{d_1-1} u_{i_2}^{d_2} \cdots u_{i_{k-1}}^{d_{k-1}} u_{i_k}^{d_k-1} \rangle$$
$$= \frac{(-1)^{l-1} (l-1)! (m-2)!}{(d_1-1)! d_2! \cdots d_{k-1}! (d_k-1)! (i_2!)^{d_2} \cdots (i_{k-1}!)^{d_{k-1}} (i_k!)^{d_k-1} (i_k+1)!} [(d_1-1)$$
$$\times \langle \underline{(u'_0 u_0)} u_0^{d_1-2} u_{i_2}^{d_2} \cdots u_{i_{k-1}}^{d_{k-1}} u_{i_k}^{d_k-1} u_{i_k+1} \rangle$$

$$+d_2\langle(\underline{u_0'u_{i_2}})u_0^{d_1-1}u_{i_2}^{d_2-1}u_{i_3}^{d_3}\cdots u_{i_{k-1}}^{d_{k-1}}u_{i_k}^{d_k-1}u_{i_k+1}\rangle$$

$$+\cdots+d_{k-1}\langle(\underline{u_0'u_{i_{k-1}}})u_0^{d_1-1}u_{i_2}^{d_2}\cdots u_{i_{k-1}}^{d_{k-1}-1}u_{i_k}^{d_k-1}u_{i_k+1}\rangle+(d_k-1)$$

$$\times\langle(\underline{u_0'u_{i_k}})u_0^{d_1-1}u_{i_2}^{d_2}\cdots u_{i_{k-1}}^{d_{k-1}}u_{i_k}^{d_k-2}u_{i_k+1}\rangle+\langle(\underline{u_0'u_{i_k+1}})u_0^{d_1-1}u_{i_2}^{d_2}\cdots u_{i_{k-1}}^{d_{k-1}}u_{i_k}^{d_k-1}\rangle]$$

$$=\frac{(-1)^{l-1}l!(m-2)!}{(d_1-1)!d_2!\cdots d_{k-1}!(d_k-1)!(i_2!)^{d_2}\cdots(i_{k-1}!)^{d_{k-1}}(i_k!)^{d_k-1}(i_k+1)!}$$

$$\times\langle\underline{u_0'}u_0^{d_1-1}u_{i_2}^{d_2}\cdots u_{i_{k-1}}^{d_{k-1}}u_{i_k}^{d_k-1}u_{i_k+1}\rangle. \tag{3.24}$$

One can remove the *bar* under u_0' in accordance with Lemma 2 so that Eq. (3.24) is totally symmetrized. Here we have assumed the validity of Eq. (3.16) for level l. The equality between Eq. (3.23) and the coefficient of Eq. (3.24) requires exactly the validity of Eq. (3.16) with $i_1=0$ at level $l+1$. We note that in the derivation of Eq. (3.24), there are some subtleties about degeneracies: Eq. (3.24) is written in the case that i_2 in the first term on the left side of Eq. (3.24) is not equal to 1; when it is one, we need to adjust the expressions. We have checked that in every case we always get the right side of Eq. (3.24).

Furthermore, we need to verify that (after the above-mentioned integration by parts) all the terms having a derivative on u_i with $i\neq 0$ on the left side of Eq. (3.22) cancel each other. Consider the terms of the form $u_{i_1}'u_{i_1}^{d_1-1}u_{i_2}^{d_2}\cdots u_{i_{k-1}}^{d_{k-1}}u_{i_k}^{d_k-1}u_{i_k+1}$, with coefficients

$$-\frac{d_1d_k}{i_k+1}C_{i_1^{d_1}i_2^{d_2}\cdots i_k^{d_k}}(m)+\frac{1}{i_1}C_{(i_1-1)i_1^{d_1-1}i_2^{d_2}\cdots i_{k-1}^{d_{k-1}}i_k^{d_k-1}(i_k+1)}(m). \tag{3.25}$$

Assuming Eq. (3.16) is true for the level-$(l+1)$ coefficient with the first index i_1-1, the vanishing of Eq. (3.25) yields the correct level-$(l+1)$ coefficient $C_{i_1^{d_1}i_2^{d_2}\cdots i_k^{d_k}}(m)$ with the first index $i_1(\neq 0)$.

Similarly, the cancellation occurs for the terms with the derivative on higher-spin currents u_{i_a} (for $i_a\neq i_1,\ i_k$) on the left-hand side of Eq. (3.22), which are of the form $u_{i_1}^{d_1}u_{i_2}^{d_2}\cdots u_{i_a}^{d_a-1}u_{i_a+1}'u_{i_{a+1}}^{d_{a+1}}\cdots u_{i_{k-1}}^{d_{k-1}}u_{i_k}^{d_k-1}u_{i_k+1}$

$$-\frac{d_k}{i_k+1}C_{i_1^{d_1}i_2^{d_2}\cdots i_a^{d_a-1}(i_a+1)i_{a+1}^{d_{a+1}}\cdots i_k^{d_k}}(m)+\frac{d_a}{i_a+1}C_{i_1^{d_1}i_2^{d_2}\cdots i_a^{d_a}i_{a+1}^{d_{a+1}}\cdots i_{k-1}^{d_{k-1}}i_k^{d_k-1}(i_k+1)}(m)=0. \tag{3.26}$$

In the same manner, we have checked the validity of Eq. (3.22) for all other partitions of the general form $u_{i_1}^{d_1}u_{i_2}^{d_2}\cdots u_{i_a}^{d_a-1}u_{i_a+1}'u_{i_{a+1}}^{d_{a+1}}\cdots u_{i_k}^{d_k}$.

With Eqs. (3.21) or (3.22) established, Eq. (3.20) reduces to

$$\oint_0\sum_l\sum_{\{i\}}C_{i_1i_2\cdots i_l}(m)\sum_{a=1}^{l}\left\langle u_{i_1}\cdots u_{i_{a-1}}\left[\sum_{k=2}^{i_a}(-1)^k\binom{i_a}{k}(u_0^{(k)}u_{i_a-k})\right.\right.$$

$$\left.\left.+\frac{(-1)^{i_a}u_0^{(i_a+2)}}{(i_a+1)(i_a+2)}\right]u_{i_{a+1}}\cdots u_{i_l}\right\rangle(z)dz=0, \tag{3.27}$$

which is equivalent to, at each level l

$$\sum_{\{j\}_l} C_{j_1^{c_1} j_2^{c_2} \cdots j_n^{c_n}}(m) \sum_{a=1}^{n} c_a \left\langle u_{j_1}^{c_1} \cdots u_{j_{a-1}}^{c_{a-1}} \left[\sum_{b=2}^{j_a} (-1)^b \binom{j_a}{b} (u_0^{(b)} u_{j_a-b}) \right] u_{j_a}^{c_a-1} u_{j_{a+1}}^{c_{a+1}} \cdots u_{j_n}^{c_n} \right\rangle$$

$$= \sum_{\{i\}_{l+1}} C_{i_1^{d_1} i_2^{d_2} \cdots i_k^{d_k}}(m) \sum_{a=1}^{k} \frac{(-1)^{i_a+1} d_a}{(i_a+1)(i_a+2)} \langle u_0^{(i_a+2)} u_{i_1}^{d_1} \cdots u_{i_{a-1}}^{d_{a-1}} u_{i_a}^{d_a-1} u_{i_{a+1}}^{d_{a+1}} \cdots u_{i_k}^{d_k} \rangle, \tag{3.28}$$

where the indices are of a similar meaning as explained below Eq. (3.22). Again we need to separate terms having the same order of derivative on one of the currents and show the cancellation among them. Notice that all terms in Eq. (3.28) are already irreducible. Consider a generic partition: $u_0^{(i_a+2)} u_{i_1}^{d_1} \cdots u_{i_{a-1}}^{d_{a-1}} u_{i_a}^{d_a-1} u_{i_{a+1}}^{d_{a+1}} \cdots u_{i_k}^{d_k}$. Only one term of such a partition appears on the symmetrized left side of Eq. (3.28), which has the coefficient

$$\frac{(-1)^{i_a+1} d_a}{(i_a+1)(i_a+2)} C_{i_1^{d_1} i_2^{d_2} \cdots i_k^{d_k}}(m). \tag{3.29}$$

But from the right side, there emerges in total k terms of the same partition

$$(-1)^{i_a} \binom{i_a+i_1+2}{i_a+2} C_{i_1^{d_1-1} (i_1+i_a+2) i_2^{d_2} \cdots i_{a-1}^{d_{a-1}} i_a^{d_a-1} i_{a+1}^{d_{a+1}} \cdots i_k^{d_k}}(m)$$

$$\times \langle (u_0^{(i_a+2)} u_{i_1}) u_{i_1}^{d_1-1} u_{i_2}^{d_2} \cdots u_{i_{a-1}}^{d_{a-1}} u_{i_a}^{d_a-1} u_{i_{a+1}}^{d_{a+1}} \cdots u_{i_k}^{d_k} \rangle$$

$$+ (-1)^{i_a} \binom{i_a+i_2+2}{i_a+2} C_{i_1^{d_1} i_2^{d_2-1} (i_2+i_a+2) i_3^{d_3} \cdots i_{a-1}^{d_{a-1}} i_a^{d_a-1} i_{a+1}^{d_{a+1}} \cdots i_k^{d_k}}(m)$$

$$\times \langle (u_0^{(i_a+2)} u_{i_2}) u_{i_1}^{d_1} u_{i_2}^{d_2-1} u_{i_3}^{d_3} \cdots u_{i_{a-1}}^{d_{a-1}} u_{i_a}^{d_a-1} u_{i_{a+1}}^{d_{a+1}} \cdots u_{i_k}^{d_k} \rangle$$

$$+ \cdots + (-1)^{i_a} \binom{2i_a+2}{i_a+2} C_{i_1^{d_1} \cdots i_{a-1}^{d_{a-1}} i_a^{d_a-2} (2i_a+2) i_{a+1}^{d_{a+1}} \cdots i_k^{d_k}}(m)$$

$$\times \langle (u_0^{(i_a+2)} u_{i_a}) u_{i_1}^{d_1} \cdots u_{i_{a-1}}^{d_{a-1}} u_{i_a}^{d_a-2} u_{i_{a+1}}^{d_{a+1}} \cdots u_{i_k}^{d_k} \rangle$$

$$+ \cdots + (-1)^{i_a} \binom{i_a+i_k+2}{i_a+2} C_{i_1^{d_1} \cdots i_{a-1}^{d_{a-1}} i_a^{d_a-1} i_{a+1}^{d_{a+1}} \cdots i_{k-1}^{d_{k-1}} i_k^{d_k-1} (i_k+i_a+2)}(m)$$

$$\times \langle (u_0^{(i_a+2)} u_{i_k}) u_{i_1}^{d_1} \cdots u_{i_{a-1}}^{d_{a-1}} u_{i_a}^{d_a-1} u_{i_{a+1}}^{d_{a+1}} \cdots u_{i_{k-1}}^{d_{k-1}} u_{i_k}^{d_k-1} \rangle$$

$$= \frac{(-1)^{i_a+l-1} (l-1)! (m-2)!}{d_1! \cdots d_{a-1}! (d_a-1)! d_{a+1}! \cdots d_k! (i_1!)^{d_1} \cdots (i_{a-1}!)^{d_{a-1}} (i_a!)^{d_a-1} (i_a+2)! (i_{a+1}!)^{d_{a+1}} \cdots (i_k!)^{d_k}}$$

$$\times [d_1 \langle (\underline{u_0^{(i_a+2)} u_{i_1}}) u_{i_1}^{d_1-1} u_{i_2}^{d_2} \cdots u_{i_{a-1}}^{d_{a-1}} u_{i_a}^{d_a-1} u_{i_{a+1}}^{d_{a+1}} \cdots u_{i_k}^{d_k} \rangle$$

$$+ d_2 \langle (\underline{u_0^{(i_a+2)} u_{i_2}}) u_{i_1}^{d_1} u_{i_2}^{d_2-1} u_{i_3}^{d_3} \cdots u_{i_{a-1}}^{d_{a-1}} u_{i_a}^{d_a-1} u_{i_{a+1}}^{d_{a+1}} \cdots u_{i_k}^{d_k} \rangle$$

$$+\cdots+(d_a-1)\langle(\underline{u_0^{(i_a+2)}u_{i_a}})u_{i_1}^{d_1}\cdots u_{i_{a-1}}^{d_{a-1}}u_{i_a}^{d_a-2}u_{i_{a+1}}^{d_{a+1}}\cdots u_{i_k}^{d_k}\rangle$$

$$+\cdots+d_k\langle(\underline{u_0^{(i_a+2)}u_{i_k}})u_{i_1}^{d_1}\cdots u_{i_{a-1}}^{d_{a-1}}u_{i_a}^{d_a-1}u_{i_{a+1}}^{d_{a+1}}\cdots u_{i_{k-1}}^{d_{k-1}}u_{i_k}^{d_k-1}\rangle]$$

$$=\frac{(-1)^{i_a+l-1}l!(m-2)!d_a}{d_1!\cdots d_k!(i_1!)^{d_1}\cdots(i_k!)^{d_k}(i_a+1)(i_a+2)}\langle\underline{u_0^{(i_a+2)}}u_{i_1}^{d_1}\cdots u_{i_{a-1}}^{d_{a-1}}u_{i_a}^{d_a-1}u_{i_{a+1}}^{d_{a+1}}\cdots u_{i_k}^{d_k}\rangle. \tag{3.30}$$

From Lemma 2 or Eq. (3.17), the right side of Eq. (3.30) is symmetrized. Its coefficient is identical to Eq. (3.29) by using Eq. (3.16), which yields Eqs. (3.27) or (3.28).

Thus we have shown that Eq. (3.16) is a solution to Eq. (3.13). In turn, it is easy to convert the above verification, particularly that from Eq. (3.23) to Eq. (3.25), into an inductive determination of the expression (3.16) for the coefficient C starting from the normalized coefficient at level $l=1$. Thus Eq. (3.16) is the unique solution to Eq. (3.13) under the Ansátz (3.15). If we had set the coefficient of the leading linear term u_{m-2} to be zero, then all other coefficients in Eq. (3.20) should vanish by induction. Q.E.D.

Now let us go beyond the Ansátz (3.15).

Proposition 5: Equation (3.16) is the only solution to Eq. (3.13) with the most general form (3.8) for charge densities.

Proof: According to Eqs. (3.4) and (3.6), any two terms of the same partition but with different permutations of currents and different orderings of local products are equal to each other, up to terms with more derivatives on currents of lower degrees (spins); in particular, any term can be expressed as the symmetrized multiple product (3.7) of the same set of currents involved plus terms with more derivatives. Therefore, we can always rewrite Eq. (3.8) as

$$H_m=\sum_l\sum_{\{i\}}C_{i_1i_2\cdots i_l}(m)\langle u_{i_1}u_{i_2}\cdots u_{i_l}\rangle+\sum_l\sum_{\{i,a\}}C_{i_1i_2\cdots i_l}^{a_1a_2\cdots a_l}(m)\langle u_{i_1}^{(a_1)}u_{i_2}^{(a_2)}\cdots u_{i_l}^{(a_l)}\rangle\equiv H_m^{(0)}+P_m, \tag{3.31}$$

with both $H_m^{(0)}$ and P_m homogeneous and of degree m, and at least one $a_k\neq 0$ in P_m. We require Eq. (3.13)

$$\left[\oint_0 H_3^{(0)}(z)dz+\oint_0 P_3(z)dz,\ \oint_0 H_m^{(0)}(w)dw+\oint_0 P_m(w)dw\right]=0. \tag{3.32}$$

Obviously, $H_3^{(0)}=u_1(z)$, and $P_3=0$ up to the total derivatives. Also note that the verification of the coefficients $C_{i_1i_2\cdots i_l}(m)$ in $H_m^{(0)}$ is independent of the presence of P_m with higher derivatives, so they are indentical to Eq. (3.16). Equation (3.32) then reduces to

$$\left[\oint_0 H_3^{(0)}(z)dz,\ \oint_0 P_m(w)dw\right]=\oint_0 dz\oint_z dw\, u_1(z)P_m(w)=0. \tag{3.33}$$

We want to show that the only solution to Eq. (3.33) is

$$\oint_0 P_m(z)dz=0. \tag{3.34}$$

Let us prove Eq. (3.34) by induction with respect to the number of derivations on currents $a=\Sigma_{j=1}^l a_j$, and the level l. First we rearrange terms by doing integration by parts so that they appear with the highest derivative on the highest-spin current being minimized, e.g.,

$u_{i_k}^{(4)}u_{i_k}^{(2)} \to u_{i_k}^{(3)}u_{i_k}^{(3)}$. After doing so, terms with different partitions now become independent to each other, or irreducible as we call it. Now let us express terms corresponding to a generic partition in the form $(u_{i_1}^{(a_1)})^{d_1}(u_{i_2}^{(a_2)})^{d_2}\cdots(u_{i_k}^{(a_k)})^{d_k}$ with $\Sigma_{j=1}^k d_j i_j = l$, $i_1 < i_2 < \cdots < i_k$ and $a_j < a_{j+1} < \cdots < a_k$ for $i_j = i_{j+1} = \cdots = i_k (j<k)$, and list them according to the ordering of (i_k, a_k, d_k), each from large to small. Consider the first nonvanishing term in the list, which has the largest values of i_k, a_k, and d_k. Denote the coefficient of this term as b. Its contribution to Eq. (3.33) reads

$$\begin{aligned}
&\oint_0 dz \oint_z dw\, b u_1(z)\langle (u_{i_1}^{(a_1)})^{d_1}(u_{i_2}^{(a_2)})^{d_2}\cdots(u_{i_k}^{(a_k)})^{d_k}\rangle(w)\\
&= -2\oint_0 dz \frac{bd_k}{i_k+1}\langle (u_{i_1}^{(a_1)})^{d_1}(u_{i_2}^{(a_2)})^{d_2}\cdots(u_{i_k}^{(a_k)})^{d_k-1}u_{i_k+1}^{(a_k+1)}\rangle(z)+\cdots\\
&= -2\oint_0 dz \frac{(-1)^{a_k+1}bd_k}{i_k+1}\langle (u_{i_1}^{(a_1)})^{d_1}(u_{i_2}^{(a_2)})^{d_2}\cdots((u_{i_k}^{(a_k)})^{d_k-1})^{(a_k+1)}u_{i_k+1}\rangle(z)\\
&\quad + \text{terms with other partitions.}
\end{aligned} \tag{3.35}$$

Note the first term on the right side has been arranged irreducible with no derivative on the highest-spin current u_{i_k+1}. Assuming Eq. (3.34) is true at $a=a$, and further when $a=a+1$ it is true at $l=l$, then there will be no other term from the left side of Eq. (3.33) matching the partition of the first term of Eq. (3.35) at $l=l+1$. It follows that b must vanish by using Eq. (3.33) and successively all terms in the list vanish, thus Eq. (3.34) is true at $a=a+1$. Note Eq. (3.34) is true at $a=0$, since in this case $P_m(z)\equiv 0$ by definition. Meanwhile, Eq. (3.34) is obviously true at $l=1$ for any a, as all P_m's are total derivatives at this level. This finishes our proof of Eq. (3.34). (Q.E.D.)

Next we proceed to prove that the commutativity (3.13) of Q_m's with Q_3 will guarantee their mutual commutativity (3.9). A similar situation happened in the literature for the search of an infinite set of commuting charges for the quantum KdV equation.[15] Essentially this is a consequence of the Jacobi identities

$$[Q_3,[Q_m,Q_n]]+[Q_m,[Q_n,Q_3]]+[Q_n,[Q_3,Q_m]]=0. \tag{3.36}$$

Proposition 6: The above constructed charges satisfy Eq. (3.9).

Proof: We recall that both Q_m and Q_n are homogeneous and of degree $m-1$ and $n-1$, respectively. Besides, the OPE's in the $\hat{W}_\infty$ algebra are homogeneous, and so is the commutator $[Q_m, Q_n]$ with degree $m+n-2$. Thus, $[Q_m, Q_n]$ must be an integral of something which is of the general form (3.8).

On one hand, substituting Eq. (3.13) into Eq. (3.36), we have immediately

$$[Q_3,[Q_m,Q_n]]=0. \tag{3.37}$$

The above-proved uniqueness of the homogeneous solution (3.15) plus (3.16) assures us that in view of Eq. (3.37), the commutator $[Q_m, Q_n]$ must be proportional to Q_{m+n-1} up to a constant factor

$$[Q_m,Q_n]=cQ_{m+n-1}. \tag{3.38}$$

We note that on the right side the charge density H_{m+n-1} is led by the linear term u_{m+n-3} and does not involve any term containing derivatives of currents.

On the other hand, as a general feature of the $\hat{W}_\infty$ algebra, the commutator between densities H_m and H_n, led by u_{m-2} and u_{n-2}, respectively, does not give rise to the desired leading u_{m+n-3} term or any term with no derivatives on currents, namely,

$$u_r(z)\cdot u_s(w)=\text{terms with derivatives on currents or in powers other than } (z-w)^{-1}. \tag{3.39}$$

This is manifest from the classical $\hat{W}_\infty$ algebra.[3] We prove that this feature survives quantization by induction. From Eq. (2.24), Eq. (3.39) is true for $r=0$, 1. Assuming it is true for $r=r$, we consider the case with r replaced by $r+1$ and write

$$u_{r+1}(z)\cdot u_s(w)=\frac{p(w)}{z-w} + \text{terms with derivatives on currents or in other powers of } (z-w)^{-1}, \tag{3.40}$$

where $p(w)$ is a purely nonderivative polynomial of currents (including u_{r+s+2}) of homogeneous degree $r+s$, appearing only in the $(z-w)^{-1}$ term. Our basic weapon is the associativity of OPE

$$u_1(z)\cdot(u_r(x)\cdot u_s(w))=(u_1(z)\cdot u_r(x))\cdot u_s(w). \tag{3.41}$$

From the left side, by the induction assumption

$$\begin{aligned} u_1(z)(u_r(x)u_s(w)) &= u_1(z)\left(\text{terms with derivatives on currents or in powers other than } \frac{1}{(x-w)}\right) \\ &= \text{terms with at least second-order derivative or two derivatives on currents} \\ &\quad \text{or in powers other than } \frac{1}{(x-w)(z-w)} \end{aligned} \tag{3.42}$$

and from the right side, using Eq. (2.24)

$$\begin{aligned} (u_1(z)u_r(x))u_s(w) &= \left(\frac{-2}{(r+1)}\frac{u'_{r+1}(x)}{(z-x)} + \text{terms with derivatives on lower spin currents} \right. \\ &\quad \left. \text{or in other powers of } \frac{1}{z-x}\right)u_s(w) \\ &= \frac{-2}{(r+1)}\frac{p'(w)}{(z-w)(x-w)} \\ &\quad + \text{terms with at least second-order derivative or two derivatives} \\ &\quad \text{on currents or in other powers of } \frac{1}{x-w} \text{ or } \frac{1}{z-w}. \end{aligned} \tag{3.43}$$

Comparing Eqs. (3.42) and (3.43), we obtain $p(w)=0$. By induction, Eq. (3.39) is true for arbitrary r.

Therefore, the constant c in Eq. (3.38) must be zero, yielding Eq. (3.9). Q.E.D.

Finally, we emphasize that in the above proofs in this section, nowhere have we used the two-boson realizations (2.8) or (2.9) of the $\hat{W}_\infty$ currents u_r.

IV. QUANTUM KP HIERARCHY

Now with the quantum $\hat{W}_\infty(p)$ algebra and an infinite set of involutive quantum $\hat{W}_\infty$ charges (at least at $p=1$) available, it is straightforward to construct a quantum version of the KP hierarchy in the Hamiltonian form (1.3).

To this end, we use the quantum $\hat{W}_\infty$ (2.19) as the quantum KP Hamiltonian structure and the densities of the quantum charges Q_m given by Eqs. (3.15) and (3.16) as corresponding Hamiltonian functions H_m. They naturally generate an infinite set of compatible flows in various times t_m $(m=1,2,...)$

$$\frac{\partial u_r}{\partial t_m}=[u_r, Q_{m+1}]. \tag{4.1}$$

Since the charges Q_m are independent of each other by construction, so are the flows they generate. Secondly, the mutual commutativity (3.9) of these quantum charges implies that they are conserved charges of the flows (4.1)

$$\frac{\partial Q_n}{\partial t_m}=[Q_n, Q_{m+1}]=0. \tag{4.2}$$

Then it is straightforward to check that the flows (4.1) are compatible, i.e.,

$$\frac{\partial^2 u_r}{\partial t_m \partial t_n}=\left[\frac{\partial u_r}{\partial t_m}, Q_{n+1}\right]=[[u_r, Q_{m+1}], Q_{n+1}]=[[u_r, Q_{n+1}], Q_{m+1}]=\frac{\partial^2 u_r}{\partial t_n \partial t_m}, \tag{4.3}$$

where Eq. (4.2) and the Jacobi identity among u_r, Q_{m+1}, and Q_{n+1} have been applied. Note Eqs. (4.1)–(4.3) give rise to the quantum counterparts of Eqs. (1.3) and (1.6). They are key features for an integrable system; especially Eq. (4.2) ensures the complete integrability of Eq. (4.1). Thus, we may call the infinite set of operator evolution equations (4.1) the $p=1$ quantum KP hierarchy, and view it as a desired quantum version of the classical KP hierarchy (1.3) or (1.1).

We present two arguments for justification of the connection to the classical KP hierarchy. First by comparaing Eq. (3.14) with Eq. (3.1) we note that at least the first three quantum charges Q_m $(m=2,3,4)$ coincide with the value at $p=1$ of the integral of the first three quantum Hamiltonians with arbitrary p, and the latter reduce to the classical KP Hamiltonians when $p\to 0$. Secondly, for arbitrary p, it is very likely, we conjecture, that there exists an infinite set of commuting quantum $\hat{W}_\infty$ charges, which reduce to the classical charges of the KP Hamiltonians (1.4) at $p=0$ and give rise to Eqs. (3.15) and (3.16) at $p=1$; indeed the charges of the densities (3.1) represent the first three of such charges. Assuming the existence of commuting quantum charges $Q_m(p)$ for all m, we propose the quantum deformation of Eq. (1.3) as

$$\frac{\partial u_r(z;p)}{\partial t_m}=[u_r(z;p), Q_{m+1}(p)]. \tag{4.4}$$

Using the explicit charge densities (3.1), we can construct the first three quantum flows of Eq. (4.4) with $m=1,2,3$. They turn out to be, by applying the OPE's (2.23)

$$\frac{\partial u_0}{\partial t_1}=(1-2p)u_0',\ \frac{\partial u_1}{\partial t_2}=(1-2p)u_1',\ \frac{\partial u_0}{\partial t_2}=(1-2p)(2u_1'+u_0''),$$

$$\frac{\partial u_1}{\partial t_2}=(2-3p)u_2'+(1-p)u_1''+\frac{(1+2p)p}{6}u_0'''+p(u_0u_0)', \tag{4.5}$$

$$\frac{\partial u_0}{\partial t_3}=3(1-2p)(u_2'+u_1'')+\frac{(3-4p-p^2)}{3}u_0'''+3p(u_0u_0)'.$$

Manipulating the last three equations, we obtain the first dynamically nontrivial quantum evolution equation for the current $u_0(z;p)$

$$6(2-3p)\frac{\partial u_0'}{\partial t_3}-9\frac{\partial^2 u_0}{\partial t_2^2}=(3-p-16p^2+18p^3)u_0''''+18(1-p)p(u_0u_0)''. \tag{4.6}$$

Rescaling $u_0\to u_0/p$ and taking $p\to 0$, this equation reduces to the classical KP equation, as expected. This equation is thus justified as a quantum deformation of the classical KP equation and the hierarchy (4.1), a quantum version of the KP hierarchy.

Next, let us discuss the field theoretical realization of the quantum KP hierarchy. Its existence is implied by the free boson realization of our quantum $\hat{W}_\infty(p)$ currents, whose charges generate the quantum KP flows. According to Eqs. (2.8) or (2.9), the KP variables $u_r(z;p)$ can be realized as functions of two bosonic currents $\bar{j}(z)$ and $j(z)$ and their derivatives, so that the flows (4.4) can be realized in terms of these currents. Alternatively, instead of performing such a reduction, we prefer to define a basic integrable hierarchy for $\bar{j}$ and j, as a quantum deformation of the classical $\bar{j}$-j hierarchy,[1,16] generated by the quantum charges

$$\frac{\partial \bar{j}}{\partial t_m}=[\bar{j},Q_{m+1}(p)],\quad \frac{\partial j}{\partial t_m}=[j,Q_{m+1}(p)], \tag{4.7}$$

with $Q_m(p)$ exactly the same as in Eqs. (4.1) or (4.4), but expressed in terms of $\bar{j}$ and j. The Hamiltonian structure of this hierarchy is directly given by the OPE's (2.7) for $\bar{j}(z)$, $j(z)$. From Eq. (4.7), one obtains all the equations of Eq. (4.4) by composition. In this way, we get a realization (or reduction) of the quantum KP hierarchy using two bosonic currents. Though the two variable hierarchy (4.7) dynamically is much less rich than the original KP hierarchy for infinitely many independent variables, the existence of such a simple realization for infinitely many KP flows is still an amazing fact. Incidentally we are reminded that the classical limit of the hierarchies (4.7) and (4.4) is recovered by rescaling $j\to j/\sqrt{p}$ and $u\to u/p$ and $[,]\to p\{,\}$, and then taking $p\to 0$.

Explicitly, the first few flows in Eq. (4.7) read

$$\frac{\partial \bar{j}}{\partial t_1}=(1-2p)\bar{j}',\quad \frac{\partial j}{\partial t_1}=(1-2p)j',$$

$$\frac{\partial \bar{j}}{\partial t_2}=(2-3p)\sqrt{p}(\bar{j}j)'+\frac{(2-3p)\sqrt{p}}{2}(\bar{j}^2)'+\frac{p\sqrt{p}}{2}(j^2)'+(1-p)\sqrt{1-2p}\bar{j}'', \tag{4.8}$$

$$\frac{\partial j}{\partial t_2}=(2-3p)\sqrt{p}(\bar{j}j)'+\frac{p\sqrt{p}}{2}(\bar{j}^2)'+\frac{(2-3p)\sqrt{p}}{2}(j^2)'-(1-p)\sqrt{1-2p}j''.$$

In the derivation we have used

$$Q_1(p)=\oint_0 (1-2p)\bar{j}j(z)dz,$$
$$Q_2(p)=\oint_0 \left[\frac{(2-3p)\sqrt{p}}{2}(\bar{j}^2j+\bar{j}j^2)+\frac{p\sqrt{p}}{6}(\bar{j}^3+j^3)-\frac{(1-p)\sqrt{1-2p}}{2}(\bar{j}j'-\bar{j}'j)\right](z)dz. \tag{4.9}$$

Now, let us discuss physical implications of the above-established quantum KP hierarchy in the conformal $SL(2,R)_k/U(1)$ model. We have taken the advantage of the quantized $SL(2,R)_k/U(1)$ model to generate the highly nontrivial quantum deformation of $\hat{W}_\infty$ in its two free boson representation, from which we have further proven the existence of an infinite set of independent and mutually commuting quantum charges, at least at level $k=p^{-1}=1$. These charges generate a huge infinite-dimensional quantum symmetry in the model, given by

$$\delta_m\bar{j}=\epsilon_m[\bar{j},Q_{m+1}],\quad \delta_m j=\epsilon_m[j,Q_{m+1}] \tag{4.10}$$

or, for the composite currents

$$\delta_m u_r=\epsilon_m[u_r,Q_{m+1}], \tag{4.11}$$

with ϵ_m the infinitesimal parameters. We can easily verify that the quantum KP flows (4.4) are in fact a set of compatible flows invariant under the symmetry transformations, namely, by using Eq. (4.2)

$$\begin{aligned}\delta_m\left(\frac{\partial u_r}{\partial t_n}-[u_r,Q_{n+1}]\right)&=\epsilon_m\left(\left[\frac{\partial u_r}{\partial t_n},Q_{m+1}\right]-[[u_r,Q_{m+1}],Q_{n+1}]\right)\\&=\epsilon_m([[u_r,Q_{n+1}],Q_{m+1}]+[[Q_{m+1},u_r],Q_{n+1}])\\&=\epsilon_m[u_r,[Q_{m+1},Q_{m+1}]]\\&=0.\end{aligned} \tag{4.12}$$

In turn, these quantum symmetry flows maintain the quantum $\hat{W}_\infty$ algebra invariant. This can be easily seen by considering the fundamental OPE's (2.7) between the basic bosonic currents $\bar{j}(z)$ and $j(z)$: Under the flows (4.7), we have infinitesimally

$$\begin{aligned}\frac{\partial}{\partial t_m}\left(\bar{j}(z)j(z')-\frac{1}{(z-z')^2}\right)&=[\bar{j}(z),Q_{m+1}]j(z')+\bar{j}(z)[j(z'),Q_{m+1}]\\&=[\bar{j}(z)j(z'),Q_{m+1}]\\&\sim\left[\frac{1}{(z-z')^2},Q_{m+1}\right]\\&=0\end{aligned} \tag{4.13}$$

and so are the OPE's between the composite quantum $\hat{W}_\infty$ currents under flows (4.4). In a certain sense, the quantum KP flows generate "canonical transformations" in the model.

In conclusion, some discussions are in order. First, we remark that the quantum KP hierarchy we have established in this article is in the Hamiltonian form (1.3), in which the complete integrability of the hierarchy appears manifest. In the classical case, the KP hierarchy is usually written in the equivalent Lax-pair form (1.1), from which all generalized Korteweg–de Vries (KdV) hierarchies can be obtained via natural reductions. It would be interesting to see if there exists a Lax-pairlike form for the quantum KP hierarchy, based on the quantum KP operator (2.17) we proposed in Sec. 2.

Related to this, an interesting problem is to see if the following expectation is true or not, i.e., our quantum KP hierarchy (4.4) would contain all known quantum KdV equations first suggested in Ref. 15, and give rise to quantum deformations of generalized classical KdV hierarchies by reduction. The completely integrable[17] quantum KdV equations have been shown to connect to perturbed conformal minimal models[18] and their charges to the vacuum singular vector of some nonunitary minimal models.[19] For the quantum KP hierarchy, while its commuting charges appear as an infinite symmetry in the noncompact conformal $SL(2,R)/U(1)$ model,[13] its direct relevance to (perhaps) perturbed coset conformal field theories remains to be clarified.

Finally, we have obtained the quantum KP hierarchy through deforming the second classical Hamiltonian structure—the nonlinear $\hat{W}_\infty$. It should be possible to obtain a quantum deformation of the classical KP hierarchy through deforming its much simpler first Hamiltonian structure—$W_{1+\infty}$. Because of its linearity, quantization of $W_{1+\infty}$ should be straightforward (either with or without a field realization). In the KP basis, the complete structure of the quantum $W_{1+\infty}$ is neatly manifested by the following OPE's:

$$u_r(z)u_s(w)=\sum_{l=0}^{r}\frac{r!}{(r-l)!}\frac{u_{r+s-l}(z)}{(z-w)^{l+1}}-\sum_{l=0}^{s}(-1)^l\frac{s!}{(s-l)!}\frac{u_{r+s-l}(w)}{(z-w)^{l+1}}+\frac{(-1)^s c r! s!}{(z-w)^{r+s+2}}+O(z-w). \tag{4.14}$$

The remaining issue is to construct an complete set of infinitely many involutive quantum charges in accordance to Eq. (4.14), in order for the associated quantum KP hierarchy to be integrable. However, we feel that the recursion relation (see, for example, Refs. 5, 3, 1) between the first and second classical KP Hamiltonian structures, or the bi-Hamiltonian structure, could not survive quantization.

Note added: After completing the work, we learned that J. Lukierski and his collaborator also attempted to quantize the KP equation (private communication).

ACKNOWLEDGMENTS

The work was supported in part by NSF Grant No. PHY-9008452.

[1] F. Yu and Y.-S. Wu, J. Math. Phys. **34**, 5851 (1993).
[2] M. Sato, RIMS Kokyuroku **439**, 30 (1981); E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, in *Proceedings of RIMS Symposium on Nonlinear Integrable Systems*, edited by M. Jimbo and T. Miwa (World Scientific, Singapore, 1983); G. Segal and G. Wilson, Publ. IHES **61**, 1 (1985).
[3] F. Yu and Y.-S. Wu, Nucl. Phys. B **373**, 713 (1992).
[4] L. Dixon, J. Lykken, and M. Peskin, Nucl. Phys. B **235**, 215 (1989).
[5] L. A. Dickey, Ann. N. Y. Acad. Sci. **491**, 131 (1987).
[6] F. Yu and Y.-S. Wu, Phys. Rev. Lett. **68**, 2996 (1992).
[7] E. Witten, Phys. Rev. D **44**, 314 (1991).
[8] Y. Watanabe, Ann. di Mat. Pura Appl. **86**, 77 (1984).
[9] C. Pope, L. Romans, and X. Shen, Phys. Lett. B **236**, 173 (1990); **242**, 401 (1990).
[10] F. Yu and Y.-S. Wu, Phys. Lett. B **263**, 220 (1991); K. Yamagishi, *ibid.* **259**, 436 (1991).
[11] A. Gerasimov, A. Marshakov, and A. Morozov, Nucl. Phys. B **328**, 664 (1989).
[12] I. Bakas and E. Kiritsis, Maryland/Berkeley/LBL preprint UCB-PTH-91/44, LBL-31213 or UMD-PP-92/37, Sept. 1991.

[13] F. Yu and Y.-S. Wu, Phys. Lett. B **294**, 177 (1992).
[14] O. Hernández, Phys. Lett. B **233**, 355 (1989).
[15] R. Sasaki and I. Yamanaka, Commun. Math. Phys. **108**, 691 (1987); Adv. Stud. Pure Math. **16**, 271 (1988).
[16] D. A. Depireux, Laval preprint LAVAL-PHY-21-92; J. M. Figueroa-O'Farrill, J. Mas, and E. Ramos, preprint BONN-HE-92/17, US-FT-92/4 or KUL-TF-92/26.
[17] B. Feigin and E. Frenkel, RIMS/Harvard preprint RIMS-827; P. Di Francesco, P. Mathieu, and D. Sénéchal, Princeton/Laval preprint PUPT-1300 or LAVAL-PHY-28/91.
[18] B. A. Kuperschmidt and P. Mathieu, Phys. Lett. B **227**, 245 (1989); T. Eguchi and S. K. Yang, *ibid.* **224**, 373 (1989); T. J. Hollywood and P. Mansfield, *ibid.* **226**, 73 (1989).
[19] T. Eguchi and S. K. Yang, Phys. Lett. B **335**, 282 (1990); A. Kuniba, T. Nakanishi, and J. Suzuki, Nucl. Phys. B **356**, 750 (1991); P. Di Francesco and P. Mathieu, Phys. Lett. B **278**, 79 (1992).